Radical Constructivism versus Piaget's Operational Constructivism in Mathematics Education

Woo, Jeong-Ho
Seoul National University
Seoul, Korea

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I. Introduction

In Greek times Plato illustrates a method of teaching mathematics according to the Socratic dialogue in his Meno. Socrates argues that he does not 'teach', but only acts as a midwife to help the child 'recollect' the true knowledge latent in the soul of the child. His dialogical method of teaching mathematics is the paradigm of the discovery method, in which the teacher assumes leadership and motivates children to inquire and learn by awakening them from ignorance and raising cognitive conflict and "conceptual discomfort" through dialogue. It may fairly be said that there are still many mathematics educators living in the pre-Socratic age, unaware of the teaching principles embodied in the Socratic method that Plato describes as guiding children from their opinions to knowledge.

One of the most fundamental principles of teaching mathematics is the principle of active learning. The awareness of this 'common sense' began to arise with the spirit of the times of Renaissance, and its theorising begins to grow as it comes to Pestalozzi through Comenius and Rousseau. To Pestalozzi, the ABC of intuition ((Zahl-Form-Aprache), which is a basic means to the building-up of thinking power, (which is the core among the three fundamental powers of being human), is not to be injected from the outside into the child, but induced by making children construct by their own activities. (Kim, Jeong-Whan), 1970)

This educational thought is connected to Dewey in the 20th century and the activistic, constructivistic methodological basis of mathematics education becomes clearer in Dewey's writing. The book ***'The psychology of number and its application to method of teaching arithmetic'*** written by Dewey and McLellan seems to be the origin of modern constructivism in mathematics education and still has a great influence on mathematics education in many elementary schools today. According to these writers, we are confronted with such problem situations as the ambiguous whole in everyday life, and our mental state of equilibrium comes to be challenged. In the process of recovering equilibrium we do measuring activities, and by just these activities we construct number as ratio, which is used as a tool for problem solving. (Dewey & McLellan, 1895) But, we ought to note here that the activism in mathematics education has undergone a revolutionary conversion in mathematical epistemology under Dewey, who denied Platonism on the ground of pragmatism. (Dewey, 1952)

In the middle of the 20th century, the constructivist position on mathematics education became clearer, and its role as an epistemological and psychological basis for the formation of mathematical concepts was developed in the operational constructivism of Piaget. And, under Piaget, mathematical epistemology recovers again its tradition of Platonism (Beth & Piaget, 1966)

The theory of Piaget as interpreted by one of his co-researchers, Inhelder, occurs as a basic principle in constructing New Mathematics curriculum in Bruner's ***'The process of education'***. Bruner supported the discipline-oriented position which emphasises the structure of knowledge, that is, the existence of fundamental concepts of the discipline and specific processes of inquiry. And, as Papert properly claims, Piaget's theory becomes an epistemological basis for the Bourbaki's mother structures as the basis for a theory of learning for 'the New Math'. (Papert, 1980)

With the eclipse of 'the New Math', so called 'radical' constructivism has developed and been advanced (von Glasersfeld, 1991) as an alternative means of mathematics education for understanding. It is related to the current thought of the times exemplified, say, as postmodern philosophy. It brings into question the faith in the existence of objective mathematical knowledge, which has been assumed in the realist/objectivist epistemology which is the unquestioned epistemological background for the traditional mathematics teaching seen in so many schools, especially in Asia. Recently, cultural relativism, an anti-Platonic epistemology which insists on the constructive, social and inter-subjective character of knowledge has constituted the dominant thought of the times. With this "spirit of the times" the problem of implementing constructivism in the classroom has been proposed as a major problem to mathematics educators, and has come to be the main subject of MERGA 17.

Radical constructivism denies traditional Platonism, ie. the reality of mathematical knowledge having universality and objectivity, and it aims to teach children so that they understand the mathematical knowledge through conversation and discussion. According to radical constructivists like von Glasersfeld and Steffe, radical constructivism is based on Piaget's theory (von Glasersfeld, 1991). A question to be raised here is whether child-centred radical constructivism can be supported by Piaget's theory, which was considered as the epistemological and psychological background of the discipline-oriented, structure-oriented 'New Math'. Moreover, it is important to note that Piaget's mathematical epistemology does not deny Platonism.

Would any mathematics teacher want to teach his/her children mathematics in such a way that they lose their faith in the objectivity of mathematical knowledge by letting the children construct mathematics more humanely? Moreover, if there were a "most deplorable gulf between the philosophy of scientists and the (relativist) philosophy of philosophers of sciences", as Freudenthal (1991, pp. 146-147) says, would it not be dangerous to make "any bond between mathematics instruction on the one hand and an alleged or assumed lack of faith in objective mathematical knowledge on the other hand"? Is the inherent inaccuracy implied in relativist epistemology productive for children?

The present paper takes the position that radical constructivism is a philosophy of mathematics education in the same vein with post modern philosophy, and is based on the American pragmatist tradition. The paper attempts to explain that radical constructivism is not generally supported by Piagetian followers in mathematics education, who trace their path to realist - not radical, but traditional - constructivism in mathematics education, and the paper also attempts to consider the situation of mathematics classrooms in Korea in relation to Piaget's operational constructivism.

II. Post-modern philosophy and radical constructivism in mathematics education

Believing that knowledge is the object existing independently of the subjects, traditional rationalists and empiricists think that reason or sensual experience is the basis for discovering knowledge. But post modern philosophers (Nietzche, Dewey, Wittgenstein, Heidegger, Feyerabend, Gadamer, Quinc, Khun, Rorty, Putnam, etc.) strongly reject the epistemic foundationalism and objectivity of knowledge, and invoke relativism, in which hermeneutical, practical and historical nature of human knowledge is emphasised. In this viewpoint, knowledge is constructed through the interaction of subjects and objects, in other words, as a result of analysing and interpreting the world under the complex operations of the multiple factors such as individual desire, motive, interest, and belief. This viewpoint of post modern philosophy denies the traditional philosophy in which it has been believed that the foundation of knowledge exists with absoluteness, universality, and certainty. By emphasising the historicity, sociality, variety, locality, contingency, and incompleteness of knowledge and its instrumental property, post modern philosophy denies foundationalism and accepts relativism as its characteristic. Cho, Hwa-Tae (1991) argues the educational implications of the post modern philosophy as follows:

Traditionally education has been considered as fostering student's ability to understand the world in a rational viewpoint by teaching the student objective knowledge about the world. But in the viewpoint of post modern philosophy, the system of knowledges we teach in the school are only social products constructed under a special viewpoint. In this viewpoint the constructive principle cannot but be taken in education, emphasising critical thought, inquiring activity, social cooperation, dialogue, subjective decision and interpretation, open examination and discussion, modification and agreement. If we accept the viewpoint of post modern philosophy, students ought to learn that the knowledges they have learned are not absolute invariant universal ones, but historical and social products formed in the context of social cultural tradition. And we also teach for them to learn that alternative viewpoints and interpretations are always possible and that it is desirable to have an open minded and flexible attitude to the viewpoints and interpretations of other people.

As von Glasersfeld (1989) has said, radical constructivism in mathematics education is a reflection of this striking philosophical current. And as Jan van den Brink (1991) said, this radical constructivism is not unrelated to the intuitionism of Brouwer. According to intuitionism, mathematics is a human activity, and cannot exist outside of the human mind. To Brouwer, mathematical thinking is a mental process of constructing the world for oneself independently of one's own experience. We construct mathematical knowledges rather than deduce the logical implications, and intuition rather than experience or logic determines the healthiness and acceptability of the ideas.

We can read the relativistic instrumentalistic and anti-Platonic view of *post modern philosophy* in the arguments of radical constructivists as follows.

> "Whatever another says or writes, you cannot but put your own subjective meanings into the words and phrases you hear.... our subjective meanings tend, of course, to become inter subjective, because we learn to modify and adapt them so that they fit the situations in which we interact with others. In this way we manage to achieve a great deal of compatibility...this means that the results of our cognitive efforts have the purpose of helping us to cope in the world of our experience, rather than the traditional goal of furnishing an 'objective' representation of a world as it might 'exist' apart from us and our experience. This attitude has much in common with the pragmatist ideas proposed by William James and John Dewey at the beginning of this century ... Such areas of relative agreement are called 'consensual domains' ... The certainty of mathematical 'facts' springs from mathematicians' observance of agreed-on ways of operating, not from the nature of an objective universe". (von Glasersfeld, 1991, pp.xiv-xv)

> "the possibility of knowledge is a function of the necessary interaction between subject and object ... knowledge, as a reflection or iconic representation of an observer-independent reality, must be replaced by knowledge as that which is in some sense 'viable' in relation to the experiential world of the knowing subject." (Konold & Johnson, 1991. p.3)

We can read the radical constructivists' interpretations of Piaget's theory to support their claims in the arguments as follows.

> "the authors ... constitute the radical wing of the constructivist front. They have taken seriously the revolutionary attitude pioneered in the 1930s by Jean Piaget, ... This attitude is characterised by the deliberate redefinition of the concept of knowledge as an *adaptive function* . In simple words, this means that the results of our cognitive efforts have the purpose of helping us to cope in the world of our experience, rather than the traditional goal of furnishing an 'objective' representation of a world as it might 'exist' apart from us and our experience. ... It is radical because it breaks with the traditional theory of knowledge" (von Glasersfeld, 1991, pp.xiv-xv)

> "In an epistemology where mathematics teaching is viewed as goal-directed interactive communication in a consensual domain of experience, mathematics learning is viewed as reflective abstraction in the context of scheme theory. In this view, mathematical knowledge is understood as co-ordinated schemes of action and operation ... using mathematics of children ... is a fundamental requirement of constructivism for mathematics education. ... determining the mathematics for children through interactive communication ... taking assimilation as the functional relation involved in learning and learning as consisting in the modifications of schemes ... is ... requirement of constructivism for mathematics education. These interiorised and reorganised schemes constituted operative mathematical concepts that are constructed by means of reflective abstraction. ... The particular modifications of a scheme could diverge in one of several directions depending on the possible learning environments encountered by the child which, in turn, are dependent on particular modifications." (Steffe, 1991, pp.178-192)

III. Piaget's Operational Constructivism and Teaching - Learning Mathematics

Piaget worked through his life to establish the biological epistemology of mathematics, being convinced of the close relationship between the structure of organisms and the logico-mathematical structure from the research on biology in his youth. (J. Piaget, 1971*)

Piaget argues that the constructive mental activities of an organism are self-regulative activities aimed at attaining equilibrium, which is one of the intrinsic characters of life. Mathematics knowledge is a form of adaptation between subjects and objects, and the development of mathematical knowledge tends to the state of complete adaptation, that is, obtaining the universal objectivity. According to Piaget, the mechanism of development of cognitive *schemes* is the same as the mechanism of organisms, and he regarded the intellectual development as the epigenetic system which has its own route, that is, the *chroeds.* The successive reconstruction of each operation ought to pass the stage corresponding to its chroed. Thus, the rate of intellectual development among individuals could be different depending on experience and environment, but the stage of development is constant.

On the assumption that there are the basic universal logico-mathematical structures common to every living subject, Piaget argues that logico-mathematical concepts are the operational schemes, which have the origin in the action schemes based on the structure of the organism, and starting from the sensory-motor schemes, reconstructed to the concrete operational schemes and then to the formal operational schemes by reflective abstraction through the general co-ordination of actions and operations.

Thus what is important for mathematical education to consider is the mechanism of the 'natural' thought by which elementary mathematical concepts are constructed through the logico-mathematical experience, which is described by Piaget as follows. (Beth & Piaget, 1966)

Logico-mathematical experience consists of the results of the actions of a subject performed upon the objects. Logico-mathematical knowledge is derived from the co-ordination of such actions by abstraction, because the properties discovered in the objects are nothing but the properties which the subject has introduced and are only ascertained from the results of the subject's actions. Logico-mathematical experience is distinguished from the physical experience related to the objects and the psychological experience which involves the subjective characteristics of actions. Logico-mathematical experience is concerned with the results of the objective and necessary actions, which will be, once interiorised, transformed into the operations.

Logico-mathematical experience is determined by the schemes of actions, which have the characteristic of co-ordination. The actions such as combining (or separating), ordering, and putting into correspondence, which form the starting point of the elementary operations of classes and relations, are the primary actions whose schemes express the general co-ordination of all actions. The intellectual behaviours at the first stage consist of the simple classifying and ordering actions and the logico-mathematical activities at the later stage are developed from them. This process of constructing the logico-mathematical knowledges is called reflective abstraction.

But what matters here are not the particular actions of individual subjects, but the most general coordinations of actions common to all subjects, originated from the schemes of actions, the roots of which are in the biological organs of the subjects, therefore referring to the universal or epistemic subject.

Thus from the beginning mathematics is not the subjective free creation of the individual subject, but the results of reconstruction of the schemes at the conscious level by reflective abstraction through the general co-ordination of relations included in the universal schemes of unconscious actions. And, the collective operations concerned with the cooperations or the social intellectual communications are the same as the operations resulted from the general co-ordination of subjective actions. The logico-mathematical operations are collective as well as personal because of the uninterrupted circularity of social contacts from an early age.

According to Piaget, the logico-mathematical operations become sophisticated by the social, educational factors, but their substances are developed to a large extent from their schemes by reflective abstraction through the coordination of collective or individual actions. He summarises the mechanism of constructing the mathematical schemes by reflective-abstractions as follows.

> "In the case of logico-mathematical abstraction, on the other hand what is given is an agglomeration of actions or operations previously made by the subject himself, with their results. In this case, abstraction consists first of taking cognisance of the existence of one of these actions or operations, that is to say, noting its possible interest, having neglected it so far; for example, the perception of correspondence was known in children, but no mathematical notice has been give before Cantor. Second, noted action are to be 'reflected' (in the physical sense of the term) by being projected into another plane ... for example, the plane of thought as opposed to that of practical action, or the plane of abstract systematization as opposed to that of concrete thought (say, algebra versus

arithmetic). Third, it has to be integrated into a new structure, which means that a new structure has to be set up, but this is only possible if two conditions are fulfilled: (a) the new structure must first of all be a reconstruction of the preceding one. ... (b) it must also, however, widen the scope of the preceding one, making it general by combining it with the elements proper to the new place of thought." (J. Piaget, 1971*, p.320)

And according to Piaget, the mechanism constructing the mathematical thought involves from the beginning the alternation of contents and forms: the trend towards progressive formalisation. Piaget says about this:

"Jusqu'ici nous assistons donc a un processus en spirale'tout reflechissement des contenus (observables) suppose l'intervention d'une forme (reflexion) et les contenus ainsi transf'er'es exigent la construction de nouvelles formes dues a' la reflexion. Il y a donc ainsi une alternance ininterrompue de reflechissements ->reflexions -> reflechissements; ot (ou) de contenus -> formes -> contenus reelabores -> nouvelles formes, etc., de domaines toujours plus larges, sans fin ni sortout de commencement absolu." (J. Piaget, 1977, p.306)

We ought to remark here that what matters is not mere co-ordination or reflection of opinions, but the conscious reconstruction of the schemes through the co-ordination and reflection of the unconscious actions or operations of the child. This point is the core of Piaget's theory, which is distinguished from other versions of constructivism, and should not be missed when we discuss the implications for mathematics education of his theory. For example, Dewey emphasized the importance of regulation of thinking and reflective thinking in the intellectual development and education, but just in the sense of "the kind of thinking that consists in turning a subject over in the mind and giving it serious and consecutive consideration." (Dewey, 1933, p.3) We do not usually expect the students to discover the concept from the facts that are presented to them or reflecting other student's opinions through discussion from nothing. It is a matter of course that to construct the mathematical concept from reflective thinking, the students already must have some basic stuff; schemes to make them see the concept. Thus students' new concepts are the ones which come from their own schemes by reflective abstraction.

The general co-ordination of actions of the epistemic subject common to all subjects has the necessity of progressive equilibration, and the universal character. And according to Piaget, the development of logico-mathematical operations consists of actualising some of the whole system of possible developments, and "this is our hypothesis, and as we see, it does not differ in all respects from that of Platonism, since it is sufficient to confer existence on these possibilities to be a Platonist." (Beth & Piaget, 1966, pp.301) But, Piaget objects to regarding the possible as the real entity so long as there has been no actualisation by an effective construction for genetic reasons.

On the other hand, Piaget isolates the three main types of structures of the subject's unconscious operations, and attempts to establish the genetic relation between such genetic structures and the Bourbaki's matrix structures. Especially, Piaget takes note of the fact that Bourbaki makes plans to derive all the other structures from the three matrix structures by differentiation or combination. That is, Piaget formalises the concrete operational structures as grouping, and emphasises the epistemological meaning of the genetic relation between the three elementary groupings; groupings of classes, relations and continuous transformations, and the Bourbaki's three matrix structures; algebraic structure, structure of order, and topological structure.

And, Piaget argues that the classifying operation and ordering operation as elementary operations and all the other logico-mathematical operations are developed from the groupings of these elementary operations, and as a paradigmatic example, he tries to show that genetically the construction of natural numbers is brought about by the progressive synthesis of groupings of classifying operation and ordering operation.

On the other hand, Piaget argues that the order of unconscious genesis of the structures of actions and operations reverses the order of conscious realisation, that is, the order of historical genesis of mathematics. And as a typical example, he tries to show the genetic relation between the order of the development of the child's spatial schemes and the theoretical development of Klein's Erlangen Programme in geometry. Piaget says based on Claparede's "law of conscious realisation" as follows.

> "Conscious realisation of a relationship is the more belated, the more primitive and automatic is its use in action (in the sense of not meeting any obstacles, conscious realisation resulting from failure at adaptation.) For example, bi-univocal correspondence, which is so elementary in acting, only entered the mathematical domain with the work of Cantor as a 'reflective' and operational concept; the group structure to be found from the sensory-motor level onwards was only isolated by Galois etc. etc. ... the inventor of these entities may very well be unaware that he is deriving them from natural thought, since he is content to construct them by using (without constructing a theory of this usage) the till then unconscious structure of his own thought." (Beth & Piaget, 1966, pp. 189-190)

So, Piaget's mathematical epistemology suggests a picture of the development of mathematics as, so called, *mental archaeology* by reflective abstraction. Because the schemes of actions and operations are deeply latent and taken as a matter of course it is so much more difficult to reflect the actions and operations on the plane of mathematical thought consciously.

According to Piaget, man comes into the world with some action schemes, and develops intellectually by differentiating and co-ordinating the schemes through interaction with environment. And, the action schemes, interiorised, become the operational schemes which are the major factors of intellectual development. The mathematical concepts are the operational schemes and gaining the insight into the mathematical concepts means to construct the related operational schemes. The logico-algebraic operations are pure operational schemes without images, and geometrical operations are the operational schemes related closely to causality. In any case, the substance of mathematics is operational scheme, and the learning of mathematical knowledges is the reconstruction of the schemes starting from the more simple and basic mathematical operational schemes (Piaget, 1974, pp.9-10)

Piaget regards the cognitive process by physical experiences and logico-mathematical experiences as the learning in a narrow sense, and together with the cognitive process by equilibration through co-ordination, decentralisation, reversibility and reciprocity as the learning in a wide sense. The schemes have the basic functions such as repetition, generalisation, differentiation, recognition, making relation between schemes or co-ordination and consist of structure (the cognitive aspect) and dynamique (the affective aspect). Motivation is nothing but the affective aspect of the schemes needing the objects for the subjects to assimilate. The need for assimilation by the functions such as repetition, generalisation, and recognition is the beginning of learning, but such a disposition for assimilation meets with resistance of the objects against assimilation and brings about the recognition of limit. This is a new source for learning and the schemes accommodate to the objects, which is to say that differentiation and coordination of schemes occurs, and the reconstructed schemes again try to assimilate the objects. Thus the schemes become differentiated and co-ordinated progressively, and develop in flexibility and variety, towards a more stabilised equilibration. This kind of 'march towards progressive equilibration' is learning. The disequilibration of schemes is occasioned by organic growth, experience, social interaction, and educational transmission. Thus, these are the factors affecting mental development, but the fundamental factor is the function of equilibration or self-regulation of the subject. (Greco et Piaget, 1974, pp21-67)

L. Montada (1978) analyses Piaget's theory from the instructional point of view and brings out the following central concepts: (a) the concepts of scheme and structure as instruments of assimilation and cognition, (b) the concept of mobility from preoperational regulation to operational reversibility, (c) the concept of equilibration as dissolution of cognitive conflict, (d) the concept of development as progressive building up of the new structure from the initial structure, (e) the concept of development as self-constructive process of the active organism. We could add to them the concepts of decentralisation, socialisation, and awareness.

According to Piaget, understanding something means the active assimilation of it to the schemes, and the cognitive development is a 'march towards equilibrium' with the environment by the cognitive functions of assimilation and accommodation. Thus the activity theory of instruction is the fundamental prerequisite for mathematical education. Piaget (1971, pp.162-163) says "This is why the active methods of educating infants succeed so much better than other methods in the teaching of abstract subjects such as arithmetic and geometry. When the child has already manipulated numbers or surfaces, as it were, before knowing them through the agency of thought, *the notion that it acquires of them subsequently consists of a genuine bringing into consciousness of already familiar schemata of action*".

Piaget emphasises using conflict, contradiction, cooperation and discussion in order to invoke the general co-ordination of schemes and its awareness by reflective abstraction.

IV. Piaget and his Followers in Mathematics Education

In the viewpoint of the traditional mathematics education, mathematics is formal systems of ready made products, and the process of mathematical discovery and the dynamic process of mathematical construction are hardly considered. It may fairly be said that the history of education for understanding is a history of pursuing the ideal of constructing knowledge in the mind of the child, (even if the expressions are different), from Greek times until now. To show that, it is enough to enumerate the names such as Plato, Descartes, Kant, Hegel, Pestalozzi, Dewey, Wertheimer, Piaget, Lakatos, Polya, Bruner, Dienes, Skemp, Freudenthal etc. ... who deny the philosophy of carving the experiences additively on the tabla rasa. In order to improve mathematics education, we have tried various approaches; the Socratic - intuitive - genetic - exemplary - discovery - heuristic - guided reinvention - all embracing activity method, instead of explanatory method. But it has always been the aspiration to improve all children's understanding of mathematics throughout.

As early as the 17th century, Descartes (1961) criticised the Euclidean synthetic scheme as suffocating the mind and emphasised the importance of analytic thinking in mathematical education. We owe to Euclid the deductivist style of mathematics and he is one of the greatest mathematics teachers in the history of mankind, but he did tend to neglect the "other half" of the mathematics thinking; analytic-heuristic thinking. And, Lakatos (1976, pp. 142-143) says properly that "Euclid has been the evil genius particularly for the history of mathematics and for the teaching of mathematics, both on the introductory and the creative levels." As Polya (1965, pp.118) says properly, "First guess, then prove - so does mathematical discovery proceed in most cases, ... the mathematics teacher has excellent opportunities to show the role of guessing in discovery and thus to impress on his students a fundamentally important attitude of mind."

According to the study of Schubring (1978), the genetic principle was brought in early 18 century in order to overcome the deficiency of such formalism that teaches mathematics as the system of ready-made knowledges developed logically, and to recapitulate in the reduced form the genesis of mathematics in the process of learning. Ever since Clairaut wrote the textbook of geometry developed by historical genetic method, up until the present, many mathematics educators have supported the genetic principle. Especially, Klein and Poincare emphasised the importance of the historical genetic principle invoking the biological genetic principle such as Haeckel's recapitualtion principle, and claimed that the history of mathematics should be the first guide of mathematics teachers. And Teoplitz, one of the disciples of Hilbert, emphasised the importance of the didactical translation of the logico/historical development of mathematics and tried to write a textbook of calculus developed according to the historical genetic principle. As recently as 1962 sixty-five prominent mathematicians in the United States and Canada, in the memorandum reacting to the New Math, supported the genetic method. (The Mathematics Teacher, March, 1962, pp.191-195). Lakatos (1976) also, claiming that the mathematics textbooks ought to be the rational reconstruction of the historical genetic process of mathematics, suggests the Socratic-genetic-heuristic approach to writing mathematics textbooks.

Piaget's theory suggests the opposite principle to the historical genetic principle in the making of the mathematics curriculum, as we could clearly read from the following arguments of Inhelder. (Bruner, 1963, pp.43-44)

> "Another matter relates particularly to the ordering of a mathematics curriculum. Often the sequence of psychological development follows more closely the axiomatic order of a subject matter than it does the historical order of development of concepts within the field .... If any special justification were

needed for teaching the structure of a subject in its proper logical or axiomatic order rather than its order of historical development, this should provide it."

According to Piaget's operational constructivism, mathematics can be more strongly connected with the human being's basic mental structure if we study more deeply the foundation of mathematical structures through its historical development. To Piaget it is desirable to attempt in early education continuously to re-form mathematics education according to the 'modern' mathematical way of thinking. Also, according to Piaget's theory, it is a natural way which is in accordance with children's mental development to grasp totality, generality and structure as simplicity, and organise textbooks by the deductive order.

An central aim in mathematics education is to overcome the mentally barren phenomenon which results from transmission of formal ready-made mathematics to students, and to develop instead a graceful and powerful mathematical thinking model, that is, "the problem of the development of 'meaning', of the 'existence' of mathematical objects" as Thom (1973) properly says. According to the historical genetic principle, the teacher could accomplish this more naturally by trying to recapitulate human being's experiences which have generated mathematics. Then could it be said that the anti-historical genetic development of school mathematics according to Piaget's theory, and the ultra-modern ways of mathematical thinking is an 'anti-didactic inversion', by the lessons of the New Math? (Freudenthal, 1973)

Piaget's view on mathematics education could be called 'a didactics of autonomous activity and operation' (eine Didatik des selbsttatigen Handels und Opercrens) as described by Inhelder (1958). Piaget (1973) suggests the following mathematics didactical principles founded on his epistemology and psychology of mathematics. First, the development of mathematical concept is the process organised by reflective abstraction through the regulation of children's activities. Thus it needs for children to gain logic-mathematical experiences by which logic-mathematical concepts are formed by reflecting children's own activities, while manipulating the concrete objects in the mathematics education of the kindergarten and early grades of elementary school. Second, because the substance of intellectual activity is operation and it is the product of regulation and internalisation of one's own activities, the mathematics education for elementary school students in the concrete operational period ought to be done by activity method. Third, a substantial improvement in mathematics education is needed in order to make children think with the 'natural' modern mathematical schemes at early stages of development. In order to accomplish this, a didactical problem is suggested which makes children's unconscious activities and structures of operations as the objects of reflection. To solve this problem, we need to consider the didactic principles such as discovery method, small group activity, awareness by appropriate discussion and intuitive method.

Inhelder suggests that Piaget's claim for necessity of logic mathematical experiences in the mathematic education of the kindergarten and early grades of elementary school could be realised as pre-curriculum, and mathematics curriculum could be constructed according to psychological-genetic sequence of mathematics rather than historical order. (J.S. Bruner, 1963)

Thom (1973) opposed very strongly the didactic position which assumes that the development of conscious awareness by the child of its unconscious activity is dominant over the emergence of the structure of operations by reflection. According to Piaget, the matrix structures of modern mathematics exist in the potential form in the schemes of child's activities and operations. It is an important educational-psychological problem whether mathematics education may be made more effective by emphasising the process of making conscious internal mechanisms of actions and thinking. But could it be compared to trying to teach the anatomic structure of leg to a child who is learning to walk, or the physiology of the digestive organs to the children who are trying to digest the overeaten food? Moreover, does the attempt to make children have the conscious knowledge about their own activities or the formal definition of the structure of their mental activities result in bad effects that spoil natural or mental activities, as when one hesitates to use language because one knows too much grammar?

As mentioned above, according to Piaget, bringing to consciousness mathematical thinking and its structure is the mechanism of learning mathematics of human beings which have appeared in the historical development of mathematics. The gradual process of awareness. If giving enough time to make embryonic mathematical thinking mature is the way to develop meaning of mathematics and to endow with existence mathematical thinking in the mental world, how long ought it to be? According to Piaget, maturity, experience, educational and social transmission broaden the possibility of cognitive development, but the realisation of the possibility depends on the self-regulation for equilibration. Thus, to Piaget, real learning is the gradual internal process of transforming the schemes. Therefore, only teaching methods which are harmonised with the mechanism of 'natural' development are desirable, and trying to make children's schemes of actions and operations conscious too early makes child's self construction impossible.

According to Piaget, the substances of mathematical activities are the operational schemes reconstructed by reflective abstraction, which starts from the coordination of the subject's activities, and the operations as means of organisation of the lower level activities become the subject matter for reflection on the next higher level. This kind of interpretation of the development of mathematical thinking ought to perhaps become a methodological basis of mathematics teaching through mathematisation.

In this vein, Aebli (1951) and Fricke (1970) developed the operational learning principle, which aims to construct operational schemes from the subject's actions which are isomorphic to the structure of the operations through internalisation and operational exercise to help the structuralisation and mobilisation of operations. But, in this 'operational didactics' the essence of mathematical thinking - reflective abstraction- is absent.

Freudenthal (1973) emphasises teaching/learning mathematics fraught with relations by the method of re-invention as progressive mathematisation through various levels of local organisation, and declared as one of the major problems of mathematics education how to stimulate reflecting on one's own physical, mental and mathematical activities. Likewise, van Hiele (1986), in his level theory of mathematical learning, also emphasises the process aspect of mathematics and the characteristics of mathematical thought. In suggesting the treatment of the inner order of thought as the subject of study in the next level, and the alternating of patterns and subjects, forms and contents, van Hiele's level theory of learning mathematics draws from Piaget, even if he is one of the famous critics against Piaget.

The Wiskobas Program of the Netherlands (Treffers, 1978), puts forward a framework for instruction theory as the gradual progressive mathematisation which has the actual phenomena as a source of mathematising, together with the structuring teaching/learning process according to micro-levels by reflection and recursion process as typified by Kilpatrick (1981), as well as the macro-structuring of the instructional courses according to Van Hiele's levels.

Viewed from this standpoint, it is necessary to identify the detailed learning levels of all the school mathematic, to study the didactical question by which phases the learning process pass from one level to the next, and how to help students make the means of organisation at the lower level become a subject matter on the next higher level. As another didactical prescription for this kind of teaching-learning mathematics, Freudenthal (1978) advocates the heterogeneous learning group comprised of pupils of different levels collaborating on one task, each on their own level. According to Freudenthal's exposition of the structure of the mathematical learning process, mathematics exercised on a lower level becomes mathematics observed on the higher level, and it is easier to observe learning processes with others than with oneself. So, this suggests learning in heterogeneous groups. And, he said that if one observes others' learning a subject matter that one has learned to master before, one objectifies this lower level activity in order to repeat it consciously even if meanwhile one has mathematised and algorithmised it.

It is very interesting to note here that Freudenthal (1973) also is known as one of the severe critics of Piaget, to the degree that Piaget (1973) himself comments about the fact. But he could not get out of the shade of Piaget's thought about the nature of mathematical knowledge, as we could read from his argument as follows. "To a large degree, mathematics is reflecting on one's own and other's physical mental and mathematical activity......This then is my fifth major problem of mathematics education: How to stimulate reflecting on one's own physical, mental and mathematical activities?" (Freudenthal, 1983) And Freudenthal (1973) also argues that the spirit of the group as the automorphism group of a structure is a general mode of actions and thinkings of all human being, and an important mode of inquiry of mathematicians, which has its origin in nature. We could not find any difference between this viewpoint and that of Piaget (1972, p.124) who says as follows. "Generally speaking, the 'group' is then the symbolic translation of certain of the fundamental characteristics of the act of intellect: the possibility of a coordination of actions, and the possibility of returns and of detours".

V. Conclusion

We do not agree that the 'radical constructivist' relativistic ideas of knowledge will cause a devastating impact on mathematics teaching. They may prove to be counter-productive, in the sense that we could foresee easily that there are many difficult problems to solve in order to practice their idea of constructing mathematics starting from the individual children's mathematics in the heterogenous classrooms. Perhaps we ought to discard the dream to find a method to solve all of the problems of teaching mathematics all at once. The radical constructivists' idea and method of teaching could make a contribution to develop the attitude and spirit of the citizen of a democratic society by emphasising conversation, communal dialogue, rationality, availability of knowledge, the creative abilities, the mathematics for slow learners especially at the primary school, and at the computer environments respecting the individual difference and level of thinking, thus diminishing the anxiety of mathematics.

In this paper we attempted to elucidate that radical constructivism is a reflection of postmodern philosophy on mathematics education and is based on the restricted interpretation of limited Piaget theory. Of course, we ought to recognise that hermeneutics belongs to human beings and anyone's interpretations of anyone's theory also belong to himself according to radical constructivists. Radical constructivists seem to fail to notice the fact that Piaget's operational constructivism does not embrace relativism on knowledge, and to the contrary, "does not differ in all respect from that of Platonism." And they also seem to ignore that the development of mathematical thinking is a process of self-awareness and reconstruction of the internal logic, that is, the schemes of epistemic subjective activity and thinking. I wonder also whether it is clearly considered by the radical constructivists that assimilation and accommodation means the variation of schemes through differentiation and coordination, from the more general and undifferentiated basic schemes to more specific coordinated ones, and that reflective abstraction is not simply reflective thinking but the reconstruction and self-awareness of the one's schemes caused from one's own reflection on the results of one's action.

According to the recent survey undertaken by Lee, In-Hyo (1991) on the real situation of the classrooms at work in the Korean high schools, teachers try to have students investigate for themselves, present and discuss, and try to invoke their internal motivation by asking thoughtful questions to them, but they fail soon to do so, due to the students' negative reactions. In general, teachers summarise systematically so called 'important contents' contained in the subject, write them on the blackboard, and try to explain it for the students, making it easy to understand by using the familiar examples. To attract the attention of students, teachers explain the contents asking routine questions or thoughtful questions and immediately giving the answers. The thought-demanding questions are asked not to derive students' thoughtful inquiry or discussion, but to help teacher himself explain more easily by letting students think for a while. They regard such explanatory lessons as asking thought-demanding questions to the students and immediately giving the answers, as the most desirable ones. Both teachers and students think that understanding sufficiently the contents in the textbook is the only thing which should be done in class. Understanding something through the inquiry learning is accompanied with the change of attitude and viewpoint, and new questions, while understanding something through such a systematic explanatory instruction brings the students to agree with the logic of the contents presented by teacher, and makes the brain clear, thus all questions disappear.

The college entrance examination is the principal offender distorting the school education in Korea, but also a major motive that makes possible even the instruction for understanding systematic knowledges. Without any interest in the subjects or the requirement to go to college, teaching the school subjects such as mathematics will be almost impossible.

However, they say, as a matter of fact, more than a half of the high school students are so called 'guests' in the mathematics class of Korea, and only a few students accept meaningfully the explanation of the teacher. This picture of mathematics classrooms is not the matter of yesterday and today as they say Euclid said that there are no royal roads in geometry. Has the real picture of mathematical education been like that from the beginning, and are there no hopes to improve mathematics education forever?

As Bruner (1972) argues, in order to put the mathematical principle in the 'mind's eye of the students', we must not teach it as a topic, but as the way of *thinking*, and we can not but let the students themselves explore and find the principle. But, in the Korean mathematics classrooms as mentioned above, teaching mathematics starting from subjective knowledges and tending to inter subjective knowledges based on the relativism of radical constructivists will be difficult to accept. Moreover education is a historical and cultural management of the nation. Radical constructivism emphasising relativity and subjectivity of knowledge and negotiation with students could not fit to the Korean traditional notion of education, 'from the mentor to the students', based on the Scripture of Confucianism.

Bruner (1968), in collaboration with Z.P. Dienes, developed a model of discovery learning which could be interpreted as a mixture of Piaget with Plato: the activity method with internalising strategy using his 'EIS' theory and Socratic dialogue. But Bruner could not regard the very core of Piaget theory: reflective abstraction and equilibration. Criticising the discovery method by Bruner, Freudenthal (1973, pp.127-130) claims that even though 8 years old children were taught factorisation of some 2nd order equations into perfect square type according to 'EIS' theory, they remained at the pre-mathematical bottom level, and the method of discovery was not adapted to raise the level of the children to the higher mathematical level by reflecting on their bottom level activities.

If we see the students' schemes of operations as 'opinions' which, Socrates says, everybody has, namely the latent knowledge that the spirit has inherently, the constructive didactics based on Piaget's theory is not different from Socrates' "obstetrics". According to Socrates, the teaching knowledge means changing the variable and unstable 'opinions' which learner already has, to more permanent and stable 'knowledge'. Typically such teaching assumes a form of refutation. Namely, teacher makes the student tell his point of view about some problem first. And then by asking successive and systematic questions about the point of view of the student, teacher awakes the student from his ignorance, gives rise to conflicts, and invokes a willingness to know. And then, again through the systematic questions, teacher makes the students accept the point of view suggested by teacher. This method may be called "obstetrics" because the teacher delivers the knowledge that is already latent in the mind of the student like a midwife. Here we admit that human beings are born with the mysterious ability to find out the principle from related facts (Lee, Hong Woo, 1979), that is, to bring into consciousness the latent schemes of operations by reflective abstraction as described by Piaget.

Individuals do not understand knowledge by conversation with a person on the same level, but by a 'self-proving' of the truth of knowledge through the learning activity engaged in with one who has higher level knowledges. The teacher on a higher level can see how his students think at the level conquered by himself a long time ago. The teacher could descend to the students' level and help their work to level up their knowledge. But, there is no method that can omit the gradual levelling up and make the students jump to the higher level at once. By presenting the irregular phenomena that cause contradiction and conflict in the learner's knowledge system, the teacher could help the students reconstruct their knowledge so that the qualitative and structural change occurs in the students' knowledge system continuously. (Eum, Tac-Dong, 1993)

The real problem which confronts mathematics teaching lies in the mental barrenness of the children learning mathematics, as the result of their habitual reception of ready made mathematical knowledge which has no real meaning to them and the meaningless repetition of the established patterns of computations. What is the intellectually honest way of teaching mathematics? What, in other words, is the way of teaching mathematics as mathematics, of developing the real meaning of school mathematics, the modes of mathematical thinking, the mathematical eyes, in the minds of the students?

Theories of modern pedagogy only suggest that teacher could guide the students' experience to discover by the subtle use of language such as Socratic dialogue, or show an example by himself, or obliquely imparting, or teach modus operandi; know-how, letting the students imitate and practice alone. (Lee, Hong-Woo, 1979)

As examined above, Piaget's operational constructivism suggests ways of humanising mathematical education by realising the idea of constructivism in mathematical education through the psychological genetic - Socratic approach. But, until now the studies for application of Piaget's theory to mathematical education were fragmentary - about limited parts of Piaget's theory. Piaget's theory, which has attempted to establish the scientific genetic epistemology is not only Piaget and his collaborators' personal works, but also the group works centred around Centre International D'epistmologic Genetique. Perhaps what is needed is a more thorough examination of Piaget's thought in its relation to the teaching of mathematics.

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Department of Mathematics Education
College of Education
Seoul National University
Kwan-ak Gu, Shin-rim Dong, San 56-1
151-742, Seoul
Korea